Using data of problem, $a=372.4$ feet. $.0009l^3+1.5708l^2=87120$. $l=221.9$ feet.

Solved in a slightly different way by V. M. Spunar.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

Edited by Dr. G. E. Wahlin, University of Illinois.

175. Proposed by H. C. FEEMSTER, A. B., Professor of Mathematics, York College, York, Nebraska.

Show (a) that $[(2n)!]/[(n+1)!n!]$ is an integer, and (b) that $[(2a)!(2b)!]/[a!b!(a+b)!]$ is an integer.

Solution by G. E. WAHLIN, Ph. D., University of Illinois.

(a) To show that $\dfrac{(2n)!}{(n+1)!(n!)^2}$ is an integer.

$$\frac{(2n)!}{(n+1)\ (n!)^2}=\frac{(2n)!}{(n+1)!\,n!}=\frac{2n(2n-1)\ (2n-2)\dots(n+1)}{(n+1)!}.$$

$I=\dfrac{2n\ (2n-1)\dots(n+1)}{n!}$ is an integer, being the quotient of n successive integers by $n!$.

For the same reason, $I'=\dfrac{(2n+1)\,(2n)\,(2n-1)\dots(n+1)}{(n+1)!}$ is an integer.

But $I'=\dfrac{2n+1}{n+1}I$; and since $2(n+1)-(2n+1)=1$,

the numerator and denominator are relatively prime in $\dfrac{2n+1}{n+1}$, and therefore since $I.\dfrac{2n+1}{n+1}$ is an integer, I must be divisible by $n+1$, or $\dfrac{I}{n+1}$ is an integer.

But $\dfrac{I}{n+1}=\dfrac{2n\ (2n-1)\dots(n+1)}{(n+1)!}=\dfrac{2n!}{(n+1)\ (n\,!)^2}.$

(b) To show that $\dfrac{(2a)!(2b)!}{a!b!\ (a+b)!}$ is an integer.

It is evident that the given expression is equal to

$$\frac{(a+1)\,(a+2)\dots.2a\,(b+1)\,(b+2)\dots.2b}{(a+b)!}\dots(1),$$

and it is therefore sufficient to show that this is an integer.

Let p be any prime less than or equal to $a+b$, and n any integer such than $p^n \leqq a+b$. Moreover, let $a=kp^n+c$ and $b=k'p^n+c'$, where c and c' are both less than p^n.

We shall determine the number of multiples of p^n in the sequence

$$a+1,\ a+2,\ \ldots,\ 2a \ldots (2).$$

Consider first the case when $c=p^n-1$. Then $a+1=(k+1)p^n$ and $2a=2kp^n+2p^n-2=(2k+1)p^n+p^n-2$, and hence $(k+1)p^n$ is, in this case, the first multiple of p^n in the sequence, and $(2k+1)p^n$ is the last. If we let v_n be the number of such multiples in the sequence, since they form an arithmetic progression, we have

$$(2k+1)p^n=(k+1)p^n+(v_n-1)p^n.$$

Solving for v_n, we get $v_n=k+1$.

If $\frac{p^n-1}{2}<c<p^n-1$, then $a+1=kp^n+c+1$, and $2a=2kp^n+2c=(2k+1)p^n+d$, $d<p^n$. Therefore, in this case again, the first and last multiples of p^n in (2) are, respectively, $(k+1)p^n$ and $(2k+1)p^n$. As before, therefore, $v_n=k+1$.

If $c \leqq \frac{p^n-1}{2}$, then $a+1=kp^n+c+1$, and $2a=2kp^n+2c$, $2c<p^n$, and in this case the first and last multiples of p^n in the sequence are, respectively, $(k+1)p^n$ and $2kp^n$. As above, we then find that $v_n=k$. We have, therefore, that

$$v_n=k+1 \text{ when } \frac{p^n-1}{2}<c\leqq p^n-1; \quad v_n=k \text{ when } c<\frac{p^n-1}{2}.$$

In the same way, letting v_n' be the number of multiples of p^n in the sequence, $(b+1)$, $(b+2)$, ..., $2b$, we find that

$$v_n'=k'+1 \text{ when } \frac{p^n-1}{2}<c'\leqq p^n-1; \quad v_n'=k' \text{ when } c'\leqq\frac{p^n-1}{2}.$$

If we now let m be the largest integral exponent for which $p^m\leqq a+b$, since all multiples of p^n are also multiples of the lower powers of p, it is not difficult to see that $v_1+v_2+\ldots+v_m$ is the power to which p enters as a factor in the product $(a+1)(a+2)$, ..., $2a$, and $v'_1+v'_2+\ldots+v'_m$ is the expo-

nent of p in the product $(b+1)(b+2)$, ..., $2b$. Hence the power of p in the numerator of (1) is

$$s=v_1+v_2+\ldots+v_m+v_1'+v_2'+\ldots v_m'.$$

We shall next turn our attention to the sequence 1, 2, 3, ..., $a+b$. In this the first multiple of p^n is p^n and as $a+b=(k+k')p^n+c+c'$ the last is $(k+k'+1)p^n$ or $(k+k')p^n$, according as $c+c'$ is greater than or less than p^n, and if we let v_n'' be the number of such multiples in this sequence we have

$$v_n''=k+k'+1 \text{ when } c+c'>p^n; \quad v_n''=k+k' \text{ when } c+c'<p^n.$$

Hence, as before, the exponent of p in the product $(a+b)!$ is

$$t=v_1''+v_2''+\ldots+v_m''.$$

But for $c+c'$ to be greater than p^n, at least one of the quantities c and c' must be greater than $\frac{p^n-1}{2}$, and it is therefore easily seen that $v_n''\leqq v_n+v_n'$ and hence $t\leqq s$.

From the condition imposed on p, this may be any prime factor of $(a+b)!$, and hence the numerator of (1) contains this prime with an exponent not less than its exponent in the denominator, and (1) is therefore an integer.

A proof of the above theorem is also found in *Die Elemente der Zahlen Theorie*, p. 37, by Bachmann.

178. Proposed by PROFESSOR L. E. DICKSON, Ph. D., The University of Chicago.

Find a formula which gives all the integral solutions prime to 5 of the congruence $y^2+z^2\equiv 0 \pmod{5^4}$.

Solution by the PROPOSER.

Since $y^2\equiv 4z^2 \pmod 5$, $y\equiv \pm 2z \pmod 5$. Hence $y=5r\pm 2z$. Then

$$y^2+z^2=5(5r^2\pm 4rz+z^2)\equiv 0 \pmod{5^4},$$
$$5r^2\pm 4rz+z^2\equiv 0 \pmod{5^3}.$$

Since z is prime to 5, the latter gives $\pm 4r+z\equiv 0 \pmod 5$, whence $r\equiv \pm z \pmod 5$. Thus $r=5s\pm z$. Substituting in $5r^2$, ..., $\equiv 0$, we get $10z^2\pm 70sz$